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Information Between Quantum Systems via POVMs

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The concepts of conditional entropy and information between subsystems of a composite quantum system are generalized to include arbitrary indirect measurements (POVMs). Some properties of those quantities differ from those of their classical counterparts; certain equalities and inequalities of classical information theory may be violated.

KEY WORDS: Quantum information; quantum conditional entropy; POVM; quantum relative entropy; entropy defect.

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In Levitin (1998, 1999), concepts of conditional entropy and information were introduced for quantum systems with respect to direct (von Neumann) measurements performed over subsystems of a composite quantum system. In this paper the concepts are generalized to include arbitrary indirect measurements (POVMs). The concepts of "conditional entropy" and "information" retain their validity for quantum systems, but their properties differ somewhat from those of their classical counterparts; specifically, some equalities and inequalities of classical information theory are in general violated.

Consider a composite quantum system consisting of two subsystems A and B, the Hilbert space $\mathcal{H}$ of the system being the tensor product, $\mathcal{H}_A \otimes \mathcal{H}_B$, of the Hilbert spaces of its two subsystems. The state of the system and the states of its subsystems are described, respectively, by the joint density matrix $\rho(A, B)$ and the marginal density matrices $\rho(A)$ and $\rho(B)$. The joint entropy of the system and the marginal entropies of the two subsystems are, respectively, $H(A, B)$, $H(A)$, and $H(B)$.

Now, let $\mathcal{M}_A = \{M_a(A)\}$ be a countable set of self-adjoint nonnegative definite operators that form a resolution of the identity (in general, non-orthogonal) in $\mathcal{H}_A$, and $\mathcal{M}_B = \{M_b(B)\}$ a similar set of operators in $\mathcal{H}_B$. The sets $\mathcal{M}_A$ and $\mathcal{M}_B$ correspond to indirect measurements (POVMs) performed respectively over

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the systems A and B. By Naimark's theorem Neimark (1940), any POVM is equivalent to a direct (von Neumann) measurement performed in an extended Hilbert space. Henceforth we will consider POVMs that correspond to measurements of a complete set of variables (represented by a complete set of orthogonal one-dimensional projectors) in the extended Hilbert space.[3]

Denote by α and β the two random variables that are the results of measurements $\mathcal{M}_A$ and $\mathcal{M}_B$. The probability distributions of α and β are

$$\begin{aligned}\Pr\{\alpha = a\} &= \mathrm{Tr}\{\rho(A)M_a(A)\},\\ \Pr\{\beta = b\} &= \mathrm{Tr}\{\rho(B)M_b(B)\},\\ \Pr\{\alpha = a, \beta = b\} &= \mathrm{Tr}\{\rho(A, B)[M_a(A) \otimes M_b(b)]\}.\end{aligned} \tag{1}$$

Lemma 1. *For any choice of $\mathcal{M}_A$ and $\mathcal{M}_B$,*

$$\begin{aligned}H(A, B) &\leq H(\alpha, \beta),\\ H(A) &\leq H(\alpha),\\ H(B) &\leq H(\beta).\end{aligned} \tag{2}$$

Proof: Inequalities (2) follow from Klein's lemma Klein (1931) by use of Naimark's theorem. □

The conditional density matrix of subsystem A given the result of a measurement performed over subsystem B can be defined for POVMs in a similar way as it is defined for von Neumann measurements, namely, following Balian (1991),

$$\rho(A|\beta = b) = \frac{\mathrm{Tr}_B\{\rho(A, B)[\mathbf{I}(A) \otimes M_b(B)]\}}{\mathrm{Tr}\{\rho(A, B)[\mathbf{I}(A) \otimes M_b(B)]\}}, \tag{3}$$

where $\mathbf{I}(A)$ is the identity operator in $\mathcal{H}_A$. Note that the denominator in (3) is just the probability $\Pr\{\beta = b\}$ for β to take on value b.

Then the conditional entropy of system A, given measurement $\mathcal{M}_B$ performed on B, is

$$H(A|\beta) = -\sum_b \mathrm{Tr}\,\{\rho(A, B)[\mathbf{I}(A) \otimes M_b(B)]\}\mathrm{Tr}\,\{\rho(A|\beta = b) \log \rho(A|\beta = b)\}. \tag{4}$$

[3] The expression "complete set of variables" is used here in exactly the same meaning as "complete set of physical quantities" (in Lifshitz, 1977, p. 5), namely, as a maximum set of simultaneously measurable quantum variables (observables). "Complete set of projectors" means that they form a resolution of the identity. Also, since sets $\mathcal{M}_A$ and $\mathcal{M}_B$ are countable, they correspond to measurements of variables with discrete spectrum.

By Klein's lemma, for any α and β (i.e., for any measurements $\mathcal{M}_A$ and $\mathcal{M}_B$ performed on A and B),

$$H(A|\beta) \leq H(\alpha|\beta), \tag{5}$$

where equality holds iff all $\rho(A|\beta = b)$ commute and $\mathcal{M}_A$ is a von Neumann measurement in the basis where all $\rho(A|\beta = b)$ are diagonal.

Since conditional entropy is meant to express the uncertainty of the state of subsystem A under the constraints imposed on it by the "best" measurement performed on subsystem B, we propose the following

Definition 1. The conditional entropy of subsystem A, conditioned by subsystem B, is

$$H(A|B) = \inf_{\mathcal{M}_B} H(A|\beta). \tag{6}$$

The following theorem states that, just as in the classical case, conditioning can only decrease the entropy of a system:

Theorem 1.

$$H(A|B) \leq H(A). \tag{7}$$

Proof:

$$H(\alpha|\beta) \leq H(\alpha),$$

$$\inf_{\mathcal{M}_B} H(\alpha|\beta) \leq H(\alpha),$$

and

$$\begin{aligned} H(A|B) &= \inf_{\mathcal{M}_B} \sum_b \Pr\{\beta = b\} \inf_{\mathcal{M}_A} H(\alpha|\beta = b) \\ &\leq \inf_{\mathcal{M}_A} \inf_{\mathcal{M}_B} H(\alpha|\beta) \leq \inf_{\mathcal{M}_A} H(\alpha) \\ &= H(A). \end{aligned}$$

□

It has been pointed out Yang (2004) that inequality (7) follows from the nonnegativity of the entropy defect Levitin (1969); Holevo (1973). Indeed,

$$\begin{aligned} H(A) - H(A|B) = \inf_{\mathcal{M}_B} [&-\mathrm{Tr}\rho(A)\log\rho(A) + \sum_b \Pr\{\beta = b\} \\ &\times \mathrm{Tr}\{\rho(A|\beta = b)\log\rho(A|\beta = b)\}], \end{aligned}$$

where the expression in brackets is a special case of the entropy defect.

Note that in classical information theory $H(\alpha, \beta) = H(\beta) + H(\alpha|\beta)$. However, this equality turns into an inequality for quantum systems:

Theorem 2.

$$H(A, B) \leq H(B) + H(A|B). \tag{8}$$

Proof: By definition (6), it suffices to prove that, for any choice of $\mathcal{M}_B$,

$$H(A, B) \leq H(B) + H(A|\beta). \tag{9}$$

It follows from Naimark's theorem that it is sufficient to prove (9) for the case when $\mathcal{M}_B$ corresponds to a complete set of orthogonal projectors (a von Neumann measurement).

Let $\{u_{ib}\}$ be the set of orthonormal eigenvectors of the conditional density matrix $\rho(A|\beta = b)$, and $\{v_b\}$ the set of orthonormal vectors corresponding to projectors $M_b(B)$. Consider a basis in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ formed by vectors $u_{i,b} \otimes v_b$. The joint density matrix $\rho(A, B) = ||\rho_{ib,i'b'}||$ in this basis has the property that $\rho_{ib,i'b} = \lambda_{ib}\delta_{ii'}$.

Let us introduce a density matrix $\rho'(A, B)$ obtained from $\rho(A, B)$ by deleting the off-diagonal elements in the basis described above, namely,

$$\rho'(A, B) = ||\rho'_{ib,i'b'}|| = ||\lambda_{ib}\delta_{ii'}\delta_{bb'}||. \tag{10}$$

For each b, the conditional density matrix is

$$\rho(A|\beta = b) = \left|\left| \frac{\lambda_{ib}\delta_{ii'}}{\sum_i \lambda_{ib}} \right|\right|.$$

Also, denote

$$\rho'(B) = \mathrm{Tr}_A\, \rho'(A, B) = ||\delta_{bb'} \sum_i \lambda_{ib}||.$$

It is readily seen that, since matrices $\rho'(A, B)$ and $\rho'(B)$ are diagonal,

$$\begin{aligned} H(A|\beta) &= -\mathrm{Tr}\, \rho'(A, B) \ln \rho'(A, B) + \mathrm{Tr}\, \rho'(B) \ln \rho'(B) \\ &= -\mathrm{Tr}\, \rho(A, B) \ln \rho'(A, B) + \mathrm{Tr}\, \rho(B) \ln \rho'(B). \end{aligned} \tag{11}$$

Consider now the quantum relative entropy (cf. Schumacher (2000); Vedral (2001)) between states $\rho(A, B)$ and $\rho'(A, B)$,

$$H(\rho(A, B)||\rho'(A, B)) = \mathrm{Tr}\, \rho(A, B) \ln \rho(A, B) - \mathrm{Tr}\, \rho(A, B) \ln \rho'(A, B), \tag{12}$$

and similarly that between $\rho(B)$ and $\rho'(B)$,

$$H(\rho(B)||\rho'(B)) = \mathrm{Tr}\, \rho(B) \ln \rho(B) - \mathrm{Tr}\, \rho(B) \ln \rho'(B). \tag{13}$$

It is well known (Vedral, 2001, p. 13, F2) that partial tracing reduces relative entropy; therefore

$$H(\rho(A,B)||\rho'(A,B)) \geq H\,(\mathrm{Tr}_A\,\rho(A,B)||\mathrm{Tr}_A\,(\rho'(A,B)) = H(\rho(B)||\rho'(B)),$$

which, by (11), (12), and (13), yields inequality (9). □

According to classical information theory, the information between the outcomes α and β of two POVMs $\mathcal{M}_A$ and $\mathcal{M}_B$ performed on subsystems A and B is

$$I(\alpha;\beta) = H(\alpha) - H(\alpha|\beta). \tag{14}$$

In the spirit of Shannon's information theory, we define the mutual information between two quantum systems as follows:

Definition 2. The information in subsystem B about subsystem A (and vice versa) is

$$I(A;B) = \sup_{\mathcal{M}_A,\mathcal{M}_B} I(\alpha;\beta). \tag{15}$$

The classical equality (14) then turns into an inequality:

Theorem 3.

$$I(A;B) \leq H(A) - H(A|B). \tag{16}$$

Proof: From the entropy defect bound Levitin (1969); Holevo (1973) it follows that for any $\mathcal{M}_B$

$$I(A;\beta) = \sup_{\mathcal{M}_A} I(\alpha;\beta) \leq H(A) - H(A|\beta), \tag{17}$$

where equality holds iff all $\rho(A|\beta = b)$ commute and $\mathcal{M}_A$ is a von Neumann measurement in the basis where all $\rho(A|\beta = b)$ are diagonal.

From (17),

$$\begin{aligned} I(A;B) &= \sup_{\mathcal{M}_A,\mathcal{M}_B} I(\alpha,\beta) \\ &\leq \sup_{\mathcal{M}_B}[H(A) - H(A|\beta)] = H(a) - \inf_{\mathcal{M}_B} H(A|\beta) \\ &= H(A) - H(A|B). \end{aligned}$$

□

The proposed measures of information and conditional entropy turn out to be useful in the analysis of correlated (in particular, entangled) quantum systems, in place of their classical counterparts.

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